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Generalized mixture operators using weighting functions: a comparative study with WA and OWA

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Abstract. In the context of multiple attribute decision making, we present an aggregation scheme based on generalized mixture operators using weighting functions and we compare it with two standard aggregation methods: weighted averaging (WA) and ordered weighted averaging (OWA). Specifically, we consider linear and quadratic weight generating functions that penalize bad attribute performances and reward good attribute performances. An illustrative example, borrowed from the literature, is used to perform the operators' comparison. We believe that this comparative study will highlight the potential and flexibility of generalized mixture operators using weighting functions that depend on attribute performances.

Keywords: fuzzy sets, weighted aggregation, linear and quadratic weight generating functions, generalized mixture operators, multiple attribute decision making.

1. Introduction

In multiple attribute decision making problems for a given number of alternatives or courses of action, one has to identify a set of relevant attributes characterizing the alternatives, elicit their relative importance and express these in some form of weight factors. The usual underlying assumption is that attributes are independent, i.e. the contribution of any individual attribute to the total rating of an alternative is independent of the other attribute values. In this paper, we assume that a numerical characterization of all relevant attributes is possible and lies within the familiar unit scale [0,1]. Moreover, as in most of the research on weighted aggregation, from the classical weighted averaging (WA) and the ordered weighted averaging (OWA) (Yager 1988) (Yager 1992; Yager 1993) to the more recent Choquet integration (over a finite domain)

(Grabisch 1995; Grabisch 1996), we assume commensurability of the various attributes within the unit scale. This means that the way attribute satisfaction values are numerically expressed in the unit interval is designed so that commensurability holds to an acceptable degree. We also assume that the relative importance of attributes can itself be expressed numerically within the unit interval.

In solving multiple attribute decision making problems many types of solution methods have been developed (see for instance (Yoon and Hwang 1995)): scoring and outranking methods, trade-off schemes, distance based methods, value and utility functions, interactive methods. The first extension of multiple attribute decision models to the fuzzy set context was proposed by Bellman and Zadeh (Bellman and Zadeh 1970). Since then most of the techniques to solve multiple attribute problems have been extended to the fuzzy domain (for good overviews see (Chen and Hwang 1992) (Zimmermann 1987) (Kickert 1978) (Ribeiro 1996)).

Scoring techniques are widely used, particularly the weighted aggregation operators based on multiple attribute value theory (Keeney and Raiffa 1976). The classical weighted aggregation is usually known in the literature by weighted averaging (WA) or simple additive weighting method (Yoon and Hwang 1995). Another important aggregation operator, within the class of weighted aggregation operators, is the ordered weighted averaging OWA (Yager 1988). In this paper we will focus on two families of generalized mixture operators (Marques Pereira and Pasi 1999; Marques Pereira 2000) using weighting functions that penalize bad attribute performances and reward good attribute performances: one family is associated with linear weight generating functions and the other family is associated with quadratic weight generating functions. This aggregation method extends weighted averaging, which is a particular case of generalized mixture operator.

The question of defining weights (i.e. weighting functions) that depend on attribute satisfaction values as in (Ribeiro and Marques Pereira 2001) is at the kernel of this work and we believe it can be relevant in some decision problems. For instance, suppose that the parents of a 5th grade student wish to select a school for their son. There are two candidate schools and the parents want to select the best one according to the following two criteria. The most important criterion for school selection is “distance”; small distance means that school is within walking distance from home, whereas large distance means that the school can be reached only by car. The second (less important) criterion is the teaching "quality" of the school.

The decisional scheme of the parents is as follows: if both schools are close from home (within walking distance) than dominance w.r.t. "distance" is much stronger than dominance w.r.t. "quality", since the difference in distance is experienced walking. On the other hand, if both schools are far from home (can be reached only by car) than dominance w.r.t. "distance" is comparable with dominance w.r.t. "quality", since the difference in distance is experienced by driving.

We model this decisional scheme by considering linear weight generating functions of the following types: *very important* (with interval [0.6, 1]) for "distance" and *important* (interval [0.4, 0.8]) for "quality". For the weighted average (WA) we consider the middle point of the weight ranges, respectively 0.8 and 0.6. Now, in Table 1 we present two possible selection cases (Case I and Case II) with attribute satisfaction values and aggregated values for each of the schools under consideration, using the two linear weight generating functions indicated before.

CASE I	Distance	Quality	RESULTS	**CASE II**	Distance	Quality	RESULTS
School 1	0.4	0.8	0.59459	School 1	0.8	0.8	0.8
School 2	0.2	1	0.63243	School 2	0.6	1	0.7951

Table 1. Example of school selection with attributes "distance" and "quality"

To clarify the use of linear weighting functions in computing the aggregated values, consider that $f_D(x_D) = 0.6 + (1-0.6)x_D$ and $f_Q(x_Q) = 0.4 + (0.8-0.4)x_Q$ are the weight generating functions for attributes "distance" and "quality", respectively. The vector $\mathbf{x} = (x_D, x_Q)$ represents the attribute satisfaction values of one school and the aggregated value of the school is given by

$$W(\mathbf{x}) = \sum_{i=D,Q} w_i(\mathbf{x})x_i$$

with weighting functions $w_i(\mathbf{x}) = f_i(x_i) / \sum_{j=D,Q} f_j(x_j)$. (1)

For instance, in Case I the aggregated value of school 1 is computed as follows,

$$Case\,1 - school\,1 = \frac{0.4*(0.6+(1-0.6)*0.4)+0.8*(0.4+(0.8-0.4)*0.8)}{(0.6+(1-0.6)*0.4)+(0.4+(0.8-0.4)*0.8} = 0.59459$$

Observing Table 1 we can see that for Case I the selected school is 2, while in Case II the selected school is 1. This simple example shows clearly the dependency of weights on the satisfaction values of attributes, rewarding higher values of attribute satisfaction and penalizing lower values. In both Cases I and II, there is a dominance of 0.2 in one attribute against an identical but reverse dominance of 0.2 in the other attribute. In Case II the dominance in the attribute "distance" wins, which is just what one would expect given that "distance" is the most important attribute. In Case I, on the other hand, it is the dominance in the attribute "quality" that wins, because in this case the "distance" satisfaction values are low and therefore the effect of the dominance that attribute is penalized. Considering the change from Case I to Case II we see that, since "distance" is the most important attribute, when its attribute satisfaction values increase from low to high the school choice changes. In other words, attributes with bad performances are penalized and attributes with good performances are rewarded, with the corresponding effect on attribute dominance.

On the other hand, if we apply either the classical weighted averaging (WA) or the ordered weighted averaging (OWA) to the school example, the selected school is always 2. This is clear in the WA case; as far as the OWA case is concerned, this is because the weight associated with attribute "distance" is the same for both schools and does not change from Case I to Case II (notice that the attribute satisfaction value for "distance" is always less or equal than that of the attribute "quality").

In summary, the main focus of this work is to illustrate the potential and flexibility of the aggregation with generalized mixture operators using weighting functions, by comparison with two other aggregation methods, the weighted averaging WA and the OWA (Yager 1988).

The paper is organized as follows. After this introductory section, section 2 provides a brief introduction to the three aggregation methods that will be compared. Section 3 presents the main concepts involved in weighted aggregation and section 4 describes the linear and quadratic weight generating functions that will be used with the generalized mixture operators. Section 5 compares and discusses the three approaches using an illustrative example, borrowed from the literature. Finally, section 6 presents the conclusions of this comparative study.

2. Weights and weighting functions

The specification of a preference ranking over a set of attributes reflects the relative importance that each attribute has in the composite. Procedures for eliciting and determining relative importance (weights) have been the focus of extensive research and discussion (Al-Kloub, Al-Shemmeri et al. 1997), (Edwards and Newman 1982) (Hobbs 1980), (Ribeiro 1996), (Stillwell, Seaver et al. 1981), (Weber and Borcherding 1993), (Yoon 1989). However, an important problem that has only been partially addressed by the research on eliciting and determining weights is the fact that weights should sometimes depend on the corresponding attribute satisfaction values. From a decision maker perspective, when considering one attribute with a high relative importance, which has a bad performance (low satisfaction value), the decisional weight of this attribute should be penalized, in order to render the given attribute less significant in the overall evaluation of the alternatives. As a result, when considering two alternatives, the dominance effect in one important attribute becomes less significant when the attribute satisfaction values are low. In the same spirit, the decisional weight of an attribute with low relative importance (weight) that has higher satisfaction values should be rewarded, thereby rendering more significant the dominance effect in that attribute. Summarizing, in some cases it may be appropriate to have weights depending on the attribute satisfaction performance values.

The introduction of weighting functions depending continuously on attribute satisfaction values produces aggregation operators with complex numerical behaviour. The monotonicity of the aggregation operator is a crucial issue (Marques Pereira and Pasi 1999; Marques Pereira 2000) which involves constraints on the derivatives of the weighted aggregation operator with respect to the various attribute satisfaction values. Naturally, the question of monotonicity is also relevant in the classical framework of numerical weights, particularly when the weighted aggregation operators have complex $\mathbf{x}$ dependencies. In this case, moreover, there is also the question of sensitivity, involving constraints on the derivatives of the aggregation operator with respect to the various weight values. Roughly speaking, while monotonicity requires that the operator increases when any of the attribute satisfaction values increases, sensitivity requires that the relative contribution of the *i-th* attribute to the value of the operator increases when the corresponding weight increases. An interesting study of monotonicity and sensitivity in the classical framework of numerical weights has been done in (Kaymac and van Nauta-Lemke 1998).

An alternative approach to modulate the relative importance of attributes depending on attribute satisfaction values was proposed by Ribeiro (Ribeiro 2000). The author introduced a linear interpolation function, called WIS (Weighting Importance and Satisfaction), which combines linguistic importance weights of attributes with attribute satisfaction values. The linguistic relative importance values are represented by fuzzy sets (Zadeh 1987) on a continuous interval support, as for instance *very_important*=[0.6, 1], and attribute satisfaction values are represented in the interval [0.1,1].

Recently, within the generalized mixture approach, two weighting functions depending on attribute satisfaction values were proposed in (Ribeiro and Marques-Pereira 2001). One is obtained from a linear weight generating function and the other is obtained from a sigmoid weight generating function. In both cases the relative importance (weights) of the attributes is provided in linguistic format, as for example {*very important, important, not important*}, and each linguistic weight has a lower and upper limit given by the decision maker. The lower and upper limits are then integrated in the weight generating functions to define their range and slope.

In this paper we develop further the generalized mixture approach by considering a quadratic extension of the linear weight generating function. The next two sections describe the main characteristics of the two weight generating functions (linear and quadratic) considered.

3. Weighted aggregation operators: WA, OWA, and generalized mixtures

In this section we briefly recall the definition and main properties of three types of weighted aggregation operators: weighted averaging (WA) operators, ordered weighted averaging (OWA) operators, and generalized mixture operators. The latter extend the classical weighted averaging operators (which remain a special case) in the sense that the classical constant weights become weighting functions defined on the aggregation domain. In other words, the weights are no longer constant but depend on the values of the aggregation variables.

The presentation of generalized mixtures follows the original ones in (Marques Pereira and Pasi 1999; Marques Pereira 2000), where mixture operators and generalized mixture operators have been introduced. Roughly speaking, mixture operators are continuous and compensative aggregation operators. In other words, they are mean operators without the monotonicity requirement. Generalized mixture operators constitute a particular class of mixture operators, in the same way as generalized mean operators (also called quasi-arithmetic means) constitute a particular class of mean operators. There is in fact a strong and interesting relation between the two "generalized' classes, as has been pointed out in (Marques Pereira and Pasi 1999).

3.1. Weighted averaging (WA) operators

This aggregation model, whose relevance stems from the axiomatic basis of multiple attribute value theory (Keeney and Raiffa 1976) together with the central role of weight determination in multiple value analysis (Weber and Borcherding 1993), is also known in the literature by simple additive weighting method (Yoon and Hwang 1995). Usually, the WA model assumes that weights are proportional to the relative value of a unit change in each attribute (Hobbs 1980).

Consider n variables in the unit interval $x_i \in [0,1]$, with $i = 1,...,n$, and an equal number of weights $w_i \geq 0$ satisfying the normalization condition $\sum_{j=1}^{n} w_j = 1$. The vectors containing the n variables and the n weights are denoted $\mathbf{x} = [x_i]$ and $\mathbf{w} = [w_i]$, respectively.

The classical weighted averaging operator is defined as $WA(\mathbf{x}) = \sum_{i=1}^{n} w_i x_i$, where $\mathbf{x} = [x_i]$ encodes the attribute satisfaction values of an alternative. The operator WA is clearly continuous in the aggregation domain $[0,1]^n$.

The weighted averaging operator $WA(\mathbf{x})$ is also compensative and monotonic, meaning that $\min(\mathbf{x}) \leq WA(\mathbf{x}) \leq \max(\mathbf{x})$ and $\mathbf{x} \geq \mathbf{y} \Rightarrow WA(\mathbf{x}) \geq WA(\mathbf{y})$, respectively. The relation $\mathbf{x} \geq \mathbf{y}$ means that $x_i \geq y_i$ for $i = 1,...,n$. Moreover, the $WA(\mathbf{x})$ operator is *linear* and the weights are *constant* in the aggregation domain.

3.2. Ordered weighted averaging (OWA) operators

The ordered weighted averaging (OWA) operators have been introduced in (Yager 1988; Yager 1992; Yager 1993). Although the original OWA scheme did not include quantifiers, in this paper we use quantified OWA operators (Yager 1996) for our comparative study. The illustrative example discussed in the final section was borrowed from the work (Yager and Kelman 1999).

The ordered weighted averaging operator is defined as $OWA(\mathbf{x}) = \sum_{i=1}^{n} w_i x_{(i)}$, where $x_{(i)}$ is the ith largest element of the vector $\mathbf{x}$. Despite the reordering mechanism, the operator WA is continuous in the aggregation domain $[0,1]^n$.

It is useful to rewrite the weighted averaging operator as $OWA(\mathbf{x}) = \sum_{i=1}^{n} w_{[i]} x_i$, where $w_{[i]}$ is the weight associated with the $i-th$ element of the vector $\mathbf{x}$, referring to the traditional OWA operator definition given above. In this formulation of the reordering mechanism it is evident that the weight associated with an element of the vector $\mathbf{x}$ changes depending on the order statistics of that element.

The ordered weighted averaging operator $OWA(\mathbf{x})$ is also compensative and monotonic, that is $\min(\mathbf{x}) \leq OWA(\mathbf{x}) \leq \max(\mathbf{x})$ and $\mathbf{x} \geq \mathbf{y} \Rightarrow OWA(\mathbf{x}) \geq OWA(\mathbf{y})$, respectively. However, the $OWA(\mathbf{x})$ operator is *non linear* and the weights are only *piecewise constant* in the aggregation domain: they are constant within comonotonicity domains but change *discontinuously* from one comonotonicity domain to another.

We recall that comonotonicity domains are cones in n dimensional aggregation space within which all vectors $\mathbf{x}$ have the same ordering signature. For instance, for $n = 3$ the two vectors $\mathbf{x} = (0.2, 0.7, 0.5)$ and $\mathbf{x'} = (0.3, 0.9, 0.8)$ have the same ordering signature ((1),(2),(3))=(2,3,1) and thus belong to the same comonotonicity domain. Instead, the vector $\mathbf{x''} = (0.6, 0.1, 0.2)$ with ordering signature ((1),(2),(3))=(1,3,2) belongs to a different comonotonicity domain.

As we shall see below, generalized mixture operators have weighting functions $w_i(\mathbf{x})$ that depend *continuously* on the aggregation variables $\mathbf{x}$. This means considering a large class of weighting functions containing, in particular, arbitrarily good approximations of the piecewise constant weighting functions used in the OWA operator. Apart from continuity, it is convenient to assume also some degree of differentiability, so that the powerful techniques of differential calculus can be applied. These techniques are in fact crucial in characterizing the general monotonicity constraints under which weighting functions effectively produce monotonic weighted aggregation operators (Marques Pereira and Pasi 1999; Marques Pereira 2000). We note that a particular class of generalized mixture operators had already been considered in (Yager and Filev 1994) but violations of monotonicity had led the authors to abandon this proposal.

Now we need to introduce a link between the OWA operator definition and its relationship with linguistic quantifiers Q, such as "Most". Given a linguistic quantifier Q, meaning for instance that "Q of the attributes should be satisfied by an acceptable solution", the weights can be determined as in (Yager 1996),

$$w_i(\mathbf{x}) = Q(\textstyle\sum_{j=1}^{i} u_{(j)}) - Q(\textstyle\sum_{j=1}^{i-1} u_{(j)}) \qquad (2)$$

where the u_i are the normalized importances of attributes, with $\sum_{j=1}^{n} u_j = 1$, and the quantifier "Most" is given by $Q(r) = r^2$ for $r \in [0,1]$.

3.3. Generalized mixture operators

Let us consider now n functions $f_i(x)$ defined in the unit interval $x \in [0,1]$, with $i = 1,...,n$. We assume that these functions are of class C^1, i.e. each function $f_i(x)$ is continuous and has a continuous derivative $f_i'(x)$ in the unit interval $x \in [0,1]$, for $i = 1,...,n$.

We also assume that the functions $f_i(x)$ are positive in the unit interval, i.e. $f_i(x) > 0$ for $x \in [0,1]$ and $i = 1,...,n$, and that their derivatives $f_i'(x)$ are non-negative in the unit interval, i.e. $f_i'(x) \geq 0$ for $x \in [0,1]$ and $i = 1,...,n$.

The generalized mixture operator $W(\mathbf{x})$ on $[0,1]^n$ generated by the n functions $f_i(x)$, with $i=1,...,n$, is defined as (Marques Pereira and Pasi 1999; Marques Pereira 2000),

$$W(\mathbf{x}) = \frac{\sum_{i=1}^{n} f_i(x_i)x_i}{\sum_{j=1}^{n} f_j(x_j)} \tag{3}$$

It is straightforward to show that $W(\mathbf{x})$ is also C^1 in the aggregation domain $[0,1]^n$, i.e. it is continuous and has continuous partial derivatives in $[0,1]^n$.

Moreover, the generalized mixture operator $W(\mathbf{x})$ can be written as a weighted average of the variables x_i, with weighting functions $w_i(\mathbf{x})$ instead of the classical constant weights w_i,

$$W(\mathbf{x}) = \sum_{i=1}^{n} w_i(\mathbf{x})x_i \qquad \text{with weighting functions} \qquad w_i(\mathbf{x}) = \frac{f_i(x_i)}{\sum_{j=1}^{n} f_j(x_j)}$$

(4)

Due to the properties of the generating functions $f_i(x)$, for $i=1,...,n$, the weighting functions are strictly positive, $w_i(\mathbf{x}) > 0$, and satisfy the normalization condition $\sum_{j=1}^{n} w_j(\mathbf{x}) = 1$.

Generalized mixture operators are compensative operators in the usual sense, but they are not necessarily monotonic. Nevertheless, interesting sufficient conditions for monotonicity can be derived, as shown below. Moreover, generalized mixture operators are non linear (even within comonotonicity domains) and their weighting functions vary continuously and smoothly in the aggregation domain.

In (Marques Pereira and Pasi 1999; Marques Pereira 2000), the authors have derived a general sufficient condition for the strict monotonicity of the generalized mixture operator $W(\mathbf{x})$, where by strict monotonicity we mean $\partial_i W(\mathbf{x}) > 0$ for $i=1,...,n$. In the unit interval, for positive generating functions with non-negative derivatives, the sufficient condition takes the simple form $f_i'(x) \le f_i(x)$, for $x \in [0,1]$ and $i = 1,...,n$. In the paper, we call it simply the monotonicity condition.

From an intuitive point of view, the monotonicity condition establishes an upper bound for the variability of the weight generating function, $f_i'(x)/f_i(x) \le 1$. In turn, this constraint limits the degree to which the weighting functions $w_i(\mathbf{x})$ depend on the aggregation variables. In other words, the weight dependence on the aggregation variables must be in some sense small (in the classical weighted averaging case it was null) so that monotonicity is not violated.

Notice that the conditions $0 \le f_i'(x) \le f_i(x) > 0$ hold independently for each $i = 1,...,n$. We shall make use of this fact in what follows by applying those conditions to a general linear or quadratic generating function $f(x)$, with only non-classical parameters regarding the weight dependence on the attribute satisfaction degrees. Later we consider the associated effective generating function $F(x) = \alpha f(x)/f(1)$, whose value for unit attribute satisfaction $F(1) = \alpha$ is given by the classical importance parameter α.

In section 4 we describe two different ways in which generalized mixture operators can model multiple attribute aggregation with weighting functions: one involves linear weight generating functions and the other involves quadratic weight generating functions.

4. Linear and quadratic weight generating functions

In this section we describe weight generating functions of the linear and quadratic types. A preliminary study of the linear case has appeared in (Ribeiro and Marques Pereira 2001). In general, as discussed in the previous section we assume that the weight generating functions, $f_i(x)$ for $x \in [0,1]$ and $i = 1,...,n$, are of class C^1 and individually satisfy three general conditions: (I) $f_i(x) > 0$, (II) $f_i'(x) \ge 0$, and (III) $f_i'(x) \le f_i(x)$. The latter is called the monotonicity condition and plays a central role in the construction.

We recall that the generalized mixture operator constructed from the generating functions is defined as $W(\mathbf{x}) = \sum_{i=1}^{n} w_i(\mathbf{x}) x_i$, where the weighting functions $w_i(\mathbf{x}) = f_i(x_i) / \sum_{j=1}^{n} f_j(x_j)$ satisfy the usual normalization condition $\sum_{j=1}^{n} w_j(\mathbf{x}) = 1$ for all $\mathbf{x} \in [0,1]^n$.

4.1. Linear weight generating functions

Consider the linear generating function $l(x) = 1 + \beta x$, with domain $x \in [0,1]$ and parametric range $0 \le \beta \le 1$. The classical case, with constant weights, corresponds to $\beta = 0$.

It is straightforward to check that the above conditions (I-III) hold in the parametric range $0 \le \beta \le 1$. The first two conditions, $l(x) > 0$ and $l'(x) \ge 0$, are clearly verified. The monotonicity condition $l'(x) \le l(x)$ can be written as $\beta(1 - x) \le 1$. Accordingly, since $(1 - x) \in [0,1]$ for $x \in [0,1]$, the monotonicity condition is also verified.

The effective generating function $L(x)$ associated with $l(x)$ is given by

$$L(x) = \alpha \frac{l(x)}{l(1)} = \alpha \frac{1 + \beta x}{1 + \beta} \tag{5}$$

with parameters $0 < \alpha \le 1$ and $0 \le \beta \le 1$.

The minimum and maximum values $L(0)$ and $L(1)$ of the effective generating function, with $0 < L(0) \le L(1) \le 1$, are given by $L(0) = \alpha / (1 + \beta)$ and $L(1) = \alpha$, respectively. Parameter $0 < \alpha \le 1$ controls the maximum value $L(1) = \alpha$ of the effective generating function, when the attribute satisfaction value is one.

In turn, parameter $0 \le \beta \le 1$ controls the ratio $L(1) / L(0) = 1 + \beta$ between the maximum and minimum values of the effective generating function. In this respect, the parametric condition $0 \le \beta \le 1$ can be written in the form $1 \le L(1) / L(0) \le 2$, meaning that the ratio $L(1) / L(0)$ is at most 2 in the linear case.

In the classical weighted averaging case, the effective generating functions have null parameter β (in this way they do not depend on attribute satisfaction) and the different constant weights are obtained by different choices of the parameter α.

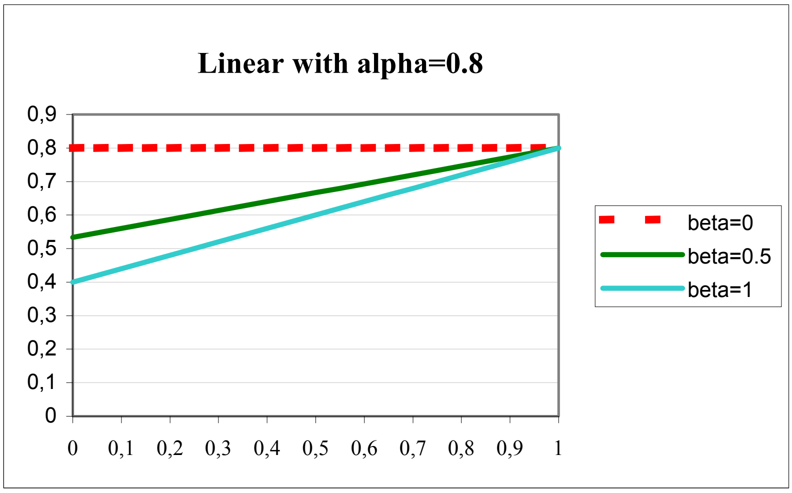

Figure 1. Linear weight generating functions

Figure 1 above shows three effective generating functions of the linear type, associated with $\alpha = 0.8$ and three different choices of parameter β. Parameter α controls the maximum value of the effective generating function, when the attribute satisfaction value is 1, while parameter β controls the ratio between the maximum and minimum values of the generating function.

4.2. Quadratic weight generating functions

In complete analogy with the linear case described previously, one can study generating functions dependending quadratically on attribute satisfaction values.

Consider thus the quadratic generating function $q(x) = 1 + \gamma x^2$, with domain $x \in [0,1]$ and parametric range $0 \leq \gamma \leq 1$. The classical case, with constant weights, corresponds to $\gamma = 0$.

As before, it is straightforward to check that conditions (I-III) hold in the parametric range $0 \leq \gamma \leq 1$. The first two conditions, $q(x) > 0$ and $q'(x) \geq 0$, are clearly verified. In turn, the monotonicity condition $q'(x) \leq q(x)$ can be written as

$\gamma(2x - x^2) \le 1$. Accordingly, since $(2x - x^2) \in [0,1]$ for $x \in [0,1]$, the monotonicity condition is also verified.

The effective generating function $Q(x)$ associated with $q(x)$ is given by

$$Q(x) = \alpha \frac{q(x)}{q(1)} = \alpha \frac{1 + \gamma x^2}{1 + \gamma} \qquad (6)$$

with parameters $0 < \alpha \le 1$ and $0 \le \gamma \le 1$.

As in the previous case, the minimum and maximum values $Q(0)$ and $Q(1)$ of the effective generating function, with $0 < Q(0) \le Q(1) \le 1$, are given by $Q(0) = \alpha/(1+\gamma)$ and $Q(1) = \alpha$, respectively. Parameter $0 < \alpha \le 1$ controls the maximum value $Q(1) = \alpha$ of the effective generating function, when the attribute satisfaction value is one.

In turn, parameter $0 \le \gamma \le 1$ controls the ratio $Q(1)/Q(0) = 1 + \gamma$ between the maximum and minimum values of the effective generating function. In this respect, the parametric condition $0 \le \gamma \le 1$ can be written in the form $1 \le Q(1)/Q(0) \le 2$, meaning that the ratio $Q(1)/Q(0)$ is at most 2 also in the quadratic case.

As before, in the classical weighted averaging case the effective generating functions have null parameter γ (in this way they do not depend on attribute satisfaction) and the different constant weights are obtained by different choices of the parameter α.

The two cases, linear and quadratic, are equivalent in most respects, particularly in the analogous role of the parameters β, γ and in the identical range (from 1 to 2) of the ratio between the maximum and minimum values of the effective generating functions.

The real difference between the linear and quadratic cases lies in the fact that the linear dependency on attribute satisfaction values is homogeneous the domain $x \in [0,1]$, whereas the quadratic dependency is weak for low attribute satisfaction values $x \in [0, \frac{1}{2}]$ and strong for high attribute satisfaction values $x \in [\frac{1}{2}, 1]$. This inhomogeneous dependency on attribute satisfaction values has the effect of emphasizing the non classical behaviour of the associated generalized mixture operators, as will be illustrated in the example presented in the last section of the paper.

Regarding the important question of the ratio between the maximum and minimum values of the effective generating functions, it is possible to significantly extend the present 1 to 2 range by considering a more general family of quadratic generating functions, containing also a linear term: $q(x) = 1 + (\beta - \gamma)x + \gamma x^2$. Roughly speaking, in this extended generating function the presence of parameter β enhances the effect of parameter γ and (with appropriate parameter tuning) the ratio $Q(1)/Q(0)$ can reach up to 2.6, approximately. The theoretical discussion of this possibility, however, is beyond the scope of this paper and will be presented elsewhere.

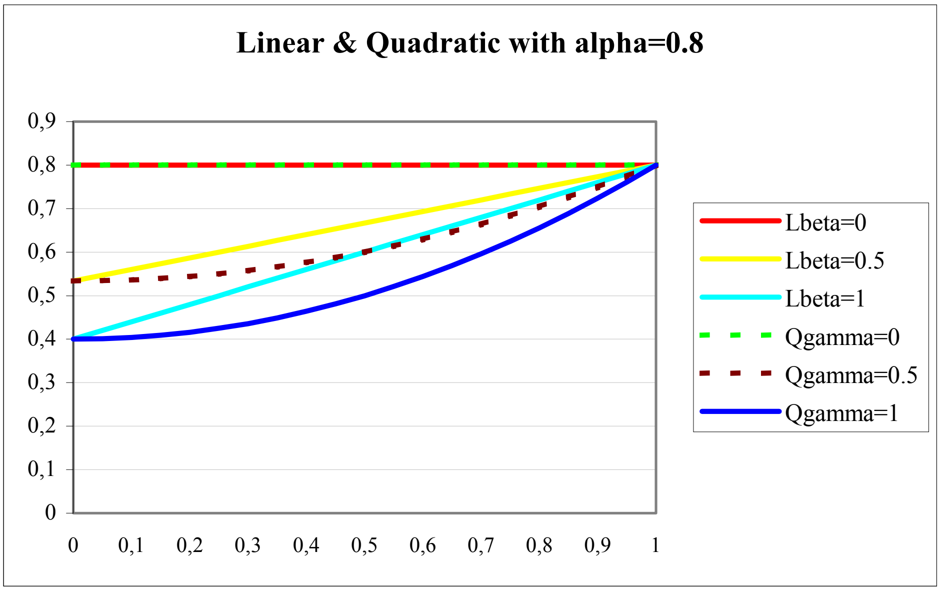

Figure 2. Linear and quadratic weight generating functions

Figure 2 above shows three effective generating functions of the quadratic type, associated with $\alpha = 0.8$ and three different choices of parameter β. The corresponding linear generating functions with $\beta = \gamma$ are also depicted. As before, parameter α controls the maximum value of the effective generating function, when the attribute satisfaction value is 1, while parameter γ controls the ratio between the maximum and minimum values of the generating function.

5. Illustrative example

This example is borrowed from Yager and Kelman (Yager and Kelman 1999) and consists in the problem of selecting the best candidate for a computer developer job. There are three candidates and their evaluation is made on the basis of six attributes. We discuss this decisional problem with the following techniques: Yager´s quantified OWA operator (Yager 1988); the classical weighted averaging operator; and the generalized mixture operators associated with our linear and quadratic weight generating functions. We compare the results obtained with the four techniques in order to illustrate the flexibility of our weight generating functions in the context of decisional problems in which dependence on attribute satisfaction is a relevant issue.

As mentioned before, there are three candidates for the computer developer job and the decision maker evaluates them with respect to six attributes. The six attributes are the computer language knowledge of: Basic, Pascal, Lisp, Prolog, C++, and Visual Basic. Moreover, the decision maker thinks that some languages are more important than others and assigning different importance values to different attributes will represent this.

For what concerns the quantified OWA method, the only difference between our discussion of the computer developer problem and the original one described in (Yager and Kelman 1999) is that we want to select the candidate with the best knowledge of *the most important languages*, instead of selecting the candidate with the best knowledge of *most languages*. For this reason, we use the linguistic quantifier "Few" instead of the linguistic quantifier "Most" originally used in (Yager and Kelman 1999). Mathematically, we represent the linguistic quantifier "Few" by the unit interval transformation $Q(r) = \sqrt{r}$ for $r \in [0,1]$.

This substitution of the linguistic quantifier "Most" with "Few" enables us to make a more meaningful comparison between our generalized mixtures and the quantified OWA operators since, in this paper, we only study the case of increasing weight generating functions. This class of generating functions produces larger weights for larger attribute satisfaction values, which is roughly speaking the average behaviour of quantified OWA operators based on the linguistic quantifier "Few". If on the other hand we were studying decreasing weight generating functions we could have kept the linguistic quantifier "Most", as in the original version of the decisional problem.

In Table 2 we show the original data for the decisional problem in question (see also (Yager and Kelman 1999)). It includes the satisfaction values of each attribute, for the three candidates, as well as the importance values of the six attributes.

Importance	1	3	2	2	5	5
Attributes	Basic	Pascal	Lisp	Prolog	C++	Visual Basic
Candidate A	0.8	0.9	0.2	0.1	0.5	0.6
Candidate B	0.2	0.7	0.1	0.3	0.7	0.5
Candidate C	0.5	0.5	0.1	0.2	0.7	0.6

Table 2. Data for the computer developer problem

The first step to solve this decisional problem is to transform the importance values into a [0,1] scale, since our effective weight generating functions are on that scale. Hence, we divide the importance of each attribute by the maximum importance value. These scaled importance values will be used throughout the discussion. Table 3 shows the original importance values and the scaled ones.

Attributes	Basic	Pascal	Lisp	Prolog	C++	Visual Basic
Importance	1	3	2	2	5	5
Scaled Importance	0.2	0.6	0.4	0.4	1	1

Table 3. Scaled importance values

The scaled importance values are used in our linear and quadratic weight generating functions to define the parameter α, which corresponds to the upper limit of the functions. Both parameters β, γ for the linear and quadratic generating functions (respectively) are set to 1, so as to have the maximal slope in both cases. Furthermore, in linguistic terms we can interpret the attribute importance values as {*very_low, low, high, very_high*} and these terms correspond to the upper limits α={0.2, 0.4, 0.6, 1}, respectively. In Figure 3 we show the linear and quadratic generating functions for the weights whose importance is *high* and *very_high* (Pascal, C++, and Visual Basic).

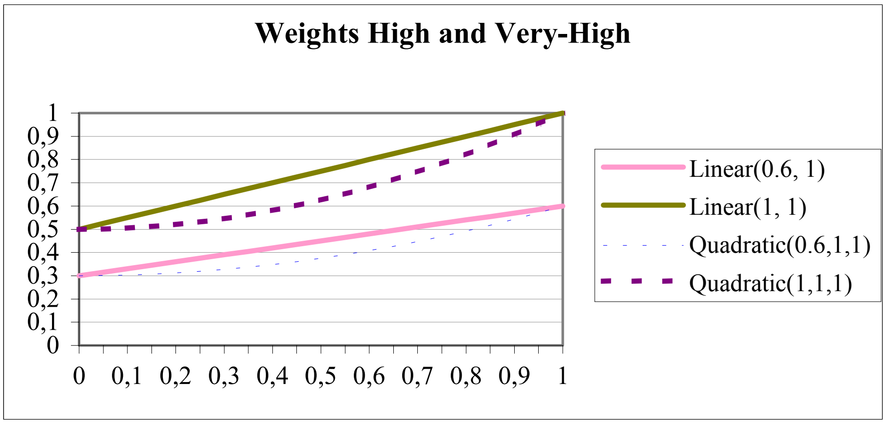

Figure 3. *high*: Linear(0.6,1), Quadratic(0.6,1); *very_high*: Linear(1,1), Quadratic(1,1).

The remaining weight generating functions used in our discussion have upper limits $\alpha = 0.2$ and $\alpha = 0.4$, corresponding to *very_low* and *low* importance values.

For what concerns the WA and OWA operators used in this example, the construction is as follows. The weights in the WA operator are given by the normalized (unit sum) importance values computed from Table 3, i.e. the weight for Pascal is 0.2/3.6 and so on. The same values of normalized importance are also used in the OWA operator construction. In this case they play the role of the u_i, for $i = 1,\ldots,n$, which enter in the definition of the OWA weights through the quantifier scheme described at the end of section 3.2.

We are now in the position to present the results using the four methods: the classical weighted averaging operator WA; Yager´s quantified OWA aggregation operator; and the generalized mixture operators associated with our linear and quadratic weight generating functions. Table 4 shows the aggregation results obtained with the four methods.

	Linear	Quadratic	WA	OWA
Candidate A	0.5732	0.5789	0.5333	0.6647
Candidate B	0.5354	0.5354	0.5056	0.5889
Candidate C	0.5332	0.5316	0.5056	0.5861

Table 4. Aggregation results

Observing Table 4 we can see that all techniques select candidate A as the best one, and that the weighted averaging operator WA fails to detect the second best candidate B. Both the OWA and our linear and quadratic weight generating functions select candidate B as the second best and candidate C as the worst candidate.

In order to illustrate the flexibility of our weight generating functions, which depend on the satisfaction values of the attributes, we made a simulation considering that candidate A had a lower attribute satisfaction value for language C++. More specifically, we wanted to see what would happen if candidate A had a satisfaction value of 0.2 instead of 0.5 for an attribute with *very_high* importance. The corresponding aggregation results are shown in table 5.

	Linear	Quadratic	WA	OWA
Candidate A	0.5084	0.5152	0.4500	0.6122
Candidate B	0.5354	0.5354	0.5056	0.5889
Candidate C	0.5332	0.5316	0.5056	0.5861

Table 5. Aggregation results using attribute C++ with 0.2 (instead of 0.5) for candidate A.

Observing the results obtained in Table 5 we can see that, now, the best candidate is B for our linear and quadratic weight generating functions, whereas the weighted averaging operator WA does not distinguish between candidate B and C, and the quantified OWA operator still thinks that candidate A is the best one.

However, if we observe the original attribute satisfaction values (Table 2), with the change of 0.2 for attribute C++ on candidate A, it is clear that candidate B performs much better (on average) in attributes with *high* and *very_high* importance, respectively Pascal, C++, and Visual Basic: with the original data (Table 2), the weighted average values of attribute satisfaction restricted to the three important attributes are A≈0.631 > B≈0.623 > C≈0.615; on the other hand, when the attribute satisfaction value of candidate A for C++ is changed from 0.5 to 0.2, the weighted average values become B≈0.623 > C≈0.615 > A≈0.515, with A as the worst candidate now. However, neither the WA nor the OWA are able to take into account this fact. This simulation clearly shows that our weight generating functions can penalize (or reward) alternatives that have lower (or higher) satisfaction values for the attributes, particularly when the attribute has a *high* or *very_high* importance.

Another important aspect to study is the comparison between the actual weighting functions of our generalized mixture operators and the OWA weights. Hence, we made a simulation that illustrates the behaviour of our weighting functions and the OWA weights when the satisfaction value of candidate A for attribute C++ increases in the interval [0.2, 0.5]. The numerical results are shown in Figure 4.

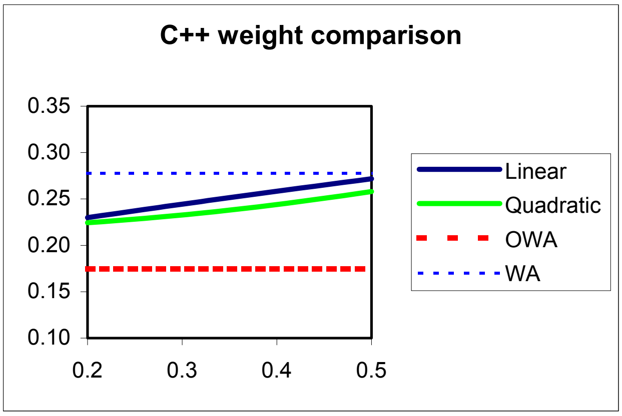

Figure 4. Comparison between our weighting functions and the OWA weights

Observing Figure 4, notice that the OWA weight is constant for the interval considered while our weighting functions reward the increase in the satisfaction value of the attribute C++. The 'linear' weighting function has a faster increase then the 'quadratic' one, as could also be seen in Figure 4. The choice between the linear and quadratic weight generating functions should be left to the decision maker because it depends on the problem, particularly on how much the decision maker wishes to penalize or reward changes in attribute satisfaction.

A final simulation was made to study violations of preferential independence (Keeney and Raiffa 1976) beyond those that occur in the OWA aggregation scheme. This means looking for preferential independence violations within comonotonicity domains, which would necessarily imply that our generalized mixtures are able to represent preference structures that can not be represented by OWA operators.

Consider a vector $\mathbf{x} = [x_i]$ of attribute satisfaction values, with $\mathbf{x} \in [0,1]^n$. As in the OWA operator definition, we denote by (i) the index value of the $i-th$ largest element of the vector $\mathbf{x}$. More precisely, we say that $((1),(2),...,(n))$ is a decreasing ordering signature associated with $\mathbf{x}$ if $x_{(1)} \geq x_{(2)} \geq ... \geq x_{(n)}$. In what follows the 'decreasing' specification will always be implicit and is therefore omitted. The ordering signature is clearly not unique when the vector $\mathbf{x}$ has equal elements. In general, we denote by $\pi(\mathbf{x})$ the set of ordering signatures associated with $\mathbf{x}$.

We say that two-attribute satisfaction vectors $\mathbf{x} = [x_i]$ and $\mathbf{y} = [y_i]$ in $[0,1]^n$ are comonotonic if the associated ordering signatures $\pi(\mathbf{x})$ and $\pi(\mathbf{y})$ coincide (as sets). Comonotonicity is thus an equivalence relation, which partitions the aggregation domain $[0,1]^n$ into mutually disjoint comonotonicity domains. Geometrically, each comonotonicity domain is a cone from the origin bounded by the limits of the aggregation domain $[0,1]^n$.

Now, consider two vectors $\mathbf{x} = [x_1,...,x_{i-1}, t, x_{i+1},..., x_n]$ and $\mathbf{y} = [y_1,..., y_{i-1}, t, y_{i+1},..., y_n]$ in $[0,1]^n$ with the same $i-th$ element. We say that an aggregation operator W on $[0,1]^n$ satisfies preferential independence with respect to the ith attribute if, given any two reduced vectors $\mathbf{x}_{\backslash i} = [x_1,..., x_{i-1}, x_{i+1},..., x_n]$ and $\mathbf{y}_{\backslash i} = [y_1,..., y_{i-1}, y_{i+1},..., y_n]$, we obtain the same relation $W(\mathbf{x}) \leq W(\mathbf{y})$ or $W(\mathbf{x}) \geq W(\mathbf{y})$ for all values of $t \in [0,1]$. We say that the aggregation operator W satisfies preferential independence if it satisfies preferential independence with respect to all attributes.

It is well known that the weighted averaging WA operator satisfies preferential independence and that the weighted averaging operator OWA only satisfies preferential independence within comonotonicity domains. We will show below that our aggregation operators, the generalized mixtures with linear and quadratic weight generating functions, do not satisfy preferential independence even within comonotonicity domains. For this reason they are able to represent preference structures not representable by OWA operators.

The example is as follows. Suppose that we have two new candidates and that they only know Pascal, Lisp and Prolog. Table 6 shows the attribute satisfaction values associated with the two new candidates, using the same attribute importance values as the ones given in Table 4. The simulation consists in increasing the satisfaction value of attribute Pascal from 0.5 to 0.7.

	Pascal	Lisp	Prolog

Candidate D	0.5	0.2	0.867
Candidate E	0.5	0.5	0.7

Table 6. Initial data for the three attributes Pascal, Lisp and Prolog.

The aggregation results are shown in Table 7. Indeed, a violation of preferential independence occurs, and within the single comonotonicity domain $x_{Lisp} < x_{Pascal} < x_{Prolog}$: candidate D is initially worse than candidate E, but ends up better than candidate E.

	Linear		OWA	
Satisfaction Pascal	Cand. D	Cand. E	Cand. D	Cand. E
0.5	0.5612	0.5624	0.6497	0.6069
0.55	0.5819	0.5826	0.6652	0.6224
0.6	0.6034	0.6036	0.6808	0.6379
0.65	0.6257	0.6253	0.6963	0.6535
0.7	0.6487	0.6478	0.7118	0.6690

Table 7. Aggregation results with increasing satisfaction of attribute Pascal (0.5-0.7)

Figure 5 shows the plot of the aggregation results, illustrating the violation of preferential independence produced by the linear weight generating function. As mentioned before, this means that our generalized mixture operators can represent preference structures that are not possible to represent with OWA operators, or for that matter with any other aggregation operator satisfying preferential independence within comonotonicity domains (as, for instance, Choquet integrals).

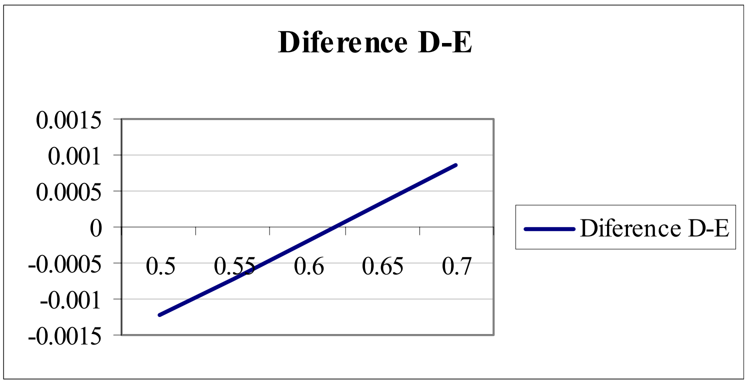

Figure 5. Preferential independence violation with linear weight generating functions

6. Conclusions

We have discussed two generalized mixture operators, associated with linear and quadratic weight generating functions, that penalize poorly satisfied attributes and reward well satisfied attributes.

We have presented an example comparing our two generalized mixture operators with the classical weighted averaging (WA) and the ordered weighted averaging (OWA) operators. The results obtained highlight the potential and flexibility of generalized mixture operators using weighting functions that depend on attribute performances.

Finally, we have shown that the generalized mixture operators associated with (say) linear weight generating functions do not satisfy preferential independence even within comonotonicity domains and therefore they can be used to represent preference structures which the OWA operators (and also the Choquet integrals) can not represent.

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