

Aggregation with generalized mixture operators using weighting functions

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Abstract

This paper regards weighted aggregation operators in multiple attribute decision making and its main goal is to investigate ways in which weights can depend on the satisfaction degrees of the various attributes (criteria). We propose and discuss two types of weighting functions that penalize poorly satisfied attributes and reward well satisfied attributes. We discuss in detail the characteristics and properties of both functions. Moreover, we present an illustrative example to clarify the behaviour of such weighting functions, comparing the results with those of standard weighted averaging.

Keywords: weighting functions, generalized mixture operators, monotonicity constraints, weighted aggregation operators, multiple attribute decision making.

1 Introduction

The main goal of this paper is to investigate ways in which the weights of weighted aggregation operators can depend on the satisfaction values of the various attributes (criteria). We propose and discuss in detail two types of weighting functions, generated by linear and quadratic weight generating functions. Both types of weighting functions penalize poorly satisfied attributes (i.e. bad attribute performances) and reward well satisfied attributes (i.e. good attribute performances). The weighted aggregation operators in our approach are special cases of the generalized mixture operators introduced in [1, 2]. Roughly

speaking, generalized mixture operators are a form of weighted averaging operators in which the classical constant weights are extended to weighting functions. In the context of generalized mixture operators the question of monotonicity can be addressed by formulating appropriate parametric constraints.

Multiple attribute decision making problems deal with ranking and selecting competing courses of action (alternatives) using attributes to evaluate the alternatives. Attributes are physical, economic or any other characteristic of the alternatives that the decision maker considers as relevant criteria for alternative selection. For example, speed and price are physical and economic characteristics of cars that can be used to rate and rank a set of alternative cars to buy. In addition, certain attributes may be more important than others for the characterization of the alternatives, in which case weights can be defined to express the relative importance of attributes.

Multiple attribute decision making problems are typically modelled by choosing a set of relevant attributes that characterize a finite number of alternatives or courses of action, and by eliciting their relative importances (weights). The usual underlying assumption is that attributes are independent, i.e. the contribution of an individual attribute to the total rating of any alternative is independent of other attribute values. Another standard assumption is that it is possible to quantify all relevant attributes within the common scale $[0,1]$. Additionally, it is also assumed that the relative importance of attributes can also be quantified within the interval $[0,1]$.

To solve multiple attribute decision making problems many types of multiple attribute techniques have been developed (see for instance [3]): scoring and outranking methods, trade-off schemes, distance based methods, value and utility functions, interactive methods. A good overview on fuzzy extensions of these techniques is described in detail in [4].

Scoring techniques are widely used, particularly the additive weighting scoring method, which is based on multiple attribute value theory [5]. This model is usually known in the literature by weighted averaging or simple additive weighting (SAW) method [3]. Usually, the SAW model assumes that weights are proportional to the relative value of a unit change in each attribute [6] and that they are normalized, $\sum_{j=1}^n w_j = 1$.

Hence, multiple attribute decision problems can be modelled as $D(A_i) = \sum_{j=1}^n x_{ij} * w_j$ where x_{ij} is the satisfaction value of attribute j in alternative A_i and w_j is the relative importance (weight) of attribute j . The best alternative is the one with the highest value in D .

The additive weighting scoring method (or SAW) is used in this paper as the aggregation method to allow us to study the characteristics and properties of two weighting functions dependent on the satisfaction values of the various attributes. The reason for selecting this weighting procedure relies on the axiomatic basis of multiple attribute value theory [5] and on the central role of weight determination in multiple value analysis [7].

Fuzzy set theory was first applied to decision making problems by Bellman and Zadeh [8]. Nowadays, the literature shows that most multiple attribute decision making techniques have been extended to the fuzzy domain (see for example [9], [4], [10] [11]).

In this paper we assume that the attribute satisfaction values are in the unit interval [0,1]. Moreover, instead of assuming the normalization condition 'a priori', we consider general positive weights within the following additive weighting scheme (also known as decision function):

$$D(A_i) = \sum_{j=1}^n x_{ij} * w_j / \sum_{k=1}^n w_k$$

where the x_{ij} are attribute satisfaction values in [0,1] and the w_j are positive weights (or importances) derived from a function dependent on attribute satisfaction values.

The specification of a preference structure over a set of attributes reflects the relative importance that each attribute has in the composite. Procedures for eliciting and determining relative importance (weights) have been the focus of extensive research and discussion [12] [6] [11] [13] [7] [14].

There are three main approaches to choosing weights [15]: indifference trade-off method, direct weighting methods, and probabilistic equivalence. Indifference trade-off methods start by asking the user to provide a measure of how much he is willing to give up in one attribute to obtain improvements in another, after which the weights are calculated by the given indifference judgments. Direct weighting methods are the most common for determining weights because they simply ask the user to give the relative importance of each attribute, or rank the attributes, after which the weights are calculated. Examples of direct weighting methods are: rank ordering [3], the ratio and swing methods introduced by Edwards and Von Winterfield [7], and the AHP Analytical Hierarchy Process method [16]. However, all these methods have some problems [15] because the notion of trade-off between attributes is not considered. Probabilistic equivalence is used to weight utility functions by expressing the probability at which the user is indifferent between an alternative that has probability p of having all attributes with best

values and probability $(1-p)$ of having attributes with worst values, when compared with the one attribute that was set as the best.

In summary, the paper is organized as follows. This first section introduces some background notions on multiple attribute decision making problems. Section 2 discusses some of the basic concepts related with the classes of weighted aggregation operators relevant to the paper. Section 3 deals with the general scheme of multiple attribute weighted aggregation using generalized mixture operators, i.e. weighting functions. Sections 4 and 5 introduce and discuss in detail *linear* weight generating functions and *quadratic* weight generating functions, respectively. Section 6 presents an illustrative example and the last section contains some concluding remarks.

2 Basic concepts and motivation

The literature on weighted aggregation operators considers essentially two fundamental ways in which weights can be introduced. In the classical weighted averaging operator (WA) weights reflect the a priori importance of the various attributes, that is the attribute satisfaction value x_i has a fixed weight w_i , for each $i=1,\dots,n$. In the more recent ordered weighted averaging operator (OWA) introduced in [17] [18] [19], weights reflect the a priori relevance of the various order statistics of the attribute satisfaction values. In other words, the *ith* largest attribute satisfaction value, denoted $x_{(i)}$, has a fixed weight w_i , for each $i=1,\dots,n$. These two fundamental ways of introducing weights are in fact the basic constructive components of a large class of weighted aggregation operators, the Choquet integrals [20] [21], containing both WA and OWA operators.

We can consider the OWA case in the classical framework of the WA operator. This can be done at the expense of introducing weighting functions $w_i(x)$, instead of the constant weights w_i , for $i=1,\dots,n$. The weighting functions $w_i(x)$ depend on the attribute satisfaction vector but, on the other hand, they are always associated with the corresponding attribute satisfaction values x_i , just as in the classical WA case. As a result of the reordering mechanism, each OWA weighting function $w_i(x)$ depends on the attribute satisfaction values in a piecewise constant manner, i.e. it is constant within each comonotonicity domain and changes *discontinuously* from one comonotonicity domain to another.

As a technical note, we recall that comonotonicity domains are cells in the multidimensional space of attribute satisfaction values within which all attribute satisfaction vectors \mathbf{x} have the same ordering

signature. With 3 attributes, the two vectors $(x_1=0.2, x_2=0.7, x_3=0.5)$ and $(x_1=0.3, x_2=0.9, x_3=0.8)$ have the same ordering signature (3,1,2) and thus belong to the same comonotonicity domain, whereas the vector $(x_1=0.6, x_2=0.1, x_3=0.2)$ with ordering signature (1,3,2) belongs to a different comonotonicity domain.

These considerations suggest an interesting new possibility, which has been recently proposed under the name generalized mixture operators [1, 2]. It consists in considering weighting functions $w_i(x)$ that depend *continuously* on the attribute satisfaction vector \mathbf{x} . This means considering a large class of weighting functions containing, in particular, arbitrarily good approximations of the piecewise constant weighting functions used in the OWA operator. Apart from continuity, it is convenient to assume also some form of differentiability, so that the powerful techniques of differential calculus can be applied. These techniques are in fact crucial in characterizing the general monotonicity constraints under which weighting functions effectively generate monotonic weighted aggregation operators [1] [2]. We note that a particular class of generalized mixture operators had already been considered in [22] but violations of monotonicity had led the authors to abandon this proposal.

The introduction of weighting functions depending continuously on attribute satisfaction values produces weighted aggregation operators with complex \mathbf{x} dependency, which is hard to study. As we have mentioned above, the central question is that of monotonicity and involves constraints on derivatives of the weighted aggregation operator with respect to the various attribute satisfaction values. Naturally, the question of monotonicity is also relevant in the classical framework of constant weights, particularly when the weighted aggregation operators have complex \mathbf{x} dependency. In this case, however, there is also the question of sensitivity, involving constraints on derivatives of the weighted aggregation operator with respect to the various numerical weights. Roughly speaking, while monotonicity requires that the operator increases when any of the attribute satisfaction values x_i increases, sensitivity requires that the relative contribution of the *ith* attribute to the value of the operator increases when the corresponding weight w_i increases. An interesting study of monotonicity and sensitivity in the classical framework of constant weights has been done in [23].

The idea of considering weighting functions that depend continuously on attribute satisfaction values (i.e. good or bad attribute performances) is supported by common sense reasoning and experience in the context of decision theory. In fact, it is not difficult to think of plausible decisional problems in which weights should, to some extent, depend on the corresponding attribute satisfaction values. From a decision maker perspective, when considering one alternative, the weight of an important attribute with a low

satisfaction value should in some cases be penalized, in order to render the given attribute less significant in the overall evaluation of the alternative. Accordingly, when considering two alternatives, the dominance effect in one important attribute would become less significant when the attribute satisfaction values are low. In the same spirit, the weight of a less important attribute with higher satisfaction values should in some cases be rewarded, thereby rendering more significant the dominance effect in that attribute. Summarizing, in some cases it may be appropriate to have weights depending on the attribute satisfaction values.

It is useful at this point to provide a simple example of a preference structure produced by a generalized mixture operator and discuss it in relation with the preference structures produced by the well known OWA operators. As mentioned above, the crucial difference between the two classes of weighted aggregation operators is that the weighting functions of OWA operators are piecewise constant within comonotonicity domains and discontinuous across comonotonicity borders, whereas the weighting functions of generalized mixture operators depend continuously on the attribute satisfaction values.

Consider the following two cases of preference comparison between two alternatives according to the satisfaction values of two attributes, as illustrated in table 1, case I and case II.

CASE I	Attribute 1	Attribute 2	CASE II	Attribute 1	Attribute 2
Alternative 1	0.5	0.8	Alternative 3	0.8	0.8
Alternative 2	0.2	1.0	Alternative 4	0.5	1.0

Table 1. A simple example

In order to compute the multiattribute score of the alternatives we will use the generalized mixture operator associated with the following weighting generating function $f(x) = (1 + x)/2$. Although the weight generating function is linear, the actual normalized weighting functions are highly non linear, $w_i(x) = f(x_i) / \sum_{j=1}^2 f(x_j)$.

The resulting generalized mixture operator is therefore $W(\mathbf{x}) = \sum_{i=1}^2 w_i(\mathbf{x}) x_i$.

The multiattribute scores can be easily computed and the results obtained are:

CASE I: Alternative 1 scores 0.6636... , Alternative 2 scores 0.7

CASE II: Alternative 3 scores 0.8 , Alternative 4 scores 0.7857...

It follows that alternative 2 is preferred in CASE I and alternative 3 is preferred in CASE II. The best alternative is indicated with a grey background in Table 1. Notice that CASE II can be obtained from

CASE I simply by adding 0.3 to the satisfaction values of attribute 1. Moreover, the satisfaction values of attribute 1 are always smaller or equal than the satisfaction values of attribute 2, in all four alternatives. This means that for these four alternatives any OWA operator, with weights (w_1, w_2) , would assign weight w_2 to the satisfaction values of attribute 1. Accordingly, the score difference between alternatives 1 and 3, given by $0.3 * w_2$ in absolute value, would be precisely the same as the score difference between alternatives 2 and 4.

The conclusion is that an OWA operator could never produce the preference reversal produced by our generalized mixture operator in this example. In other words, the central question is that OWA operators are linear within comonotonicity domains (as in this example) whereas generalized mixture operators are non linear even *within* comonotonicity domains. We believe that this property of generalized mixture operators, together with their natural geometrical analogy with OWA operators, renders them an interesting object of study.

A different proposal to modulate the weight of attributes depending on attribute satisfaction values was made by Ribeiro [24]. The author proposed a linear interpolation function, called WIS (Weighting Importance and Satisfaction), which combines linguistic importance (weight) of attributes with attribute satisfaction values. The linguistic importance (weights) are represented by linguistic labels [25] on a continuous interval support, as for instance *very_important*=[0.6, 1], and attribute satisfaction values are represented by values in the restricted interval [0.1,1]. The lower limit 0.1 represents the minimum satisfaction value required by this method. Moreover, the WIS function is used as the typical term in the total rating process, i.e. it substitutes the combination operation $x_{ij} * w_j$ of the decision function, where the x_{ij} are the attribute satisfaction values and the w_j are the relative importance (weights) of the attributes, as assigned by the decision maker. The combined values $WIS(x_{ij}, w_j)$ are then added and the result is normalized by dividing by the sum of the effective weights, given by the ratios $WIS(x_{ij}, w_j)/x_{ij}$. In this way the final rating for each alternative is obtained. There are two main problems with this proposal. First, it requires a minimum level of attribute satisfaction, in this case 0.1 (threshold α), since the function behaves incoherently below that level. Second, the WIS function is used as a combination operator instead of being used for defining directly the effective weights (WIS/X) on the basis of attribute satisfaction values, as required by the classical decision function. Clearly, the WIS function needs further improvements to support its claims.

More recently, two weighting functions based on attribute satisfaction values were proposed [26] in the context of the generalized mixture approach. These do not have the problems found in the previous proposal (WIS). One is generated by a linear weight generating function and the other is generated by a sigmoid weight generating function. In both cases, the relative importance (weight) of attributes is provided in linguistic format, as for example $\{very\ important, important, not\ important\}$, and each linguistic weight has a lower and upper limit given by the decision maker. The lower and upper limits are then integrated in the weighting functions to define their range and slope.

3 Generalized mixture operators and monotonicity

In this section we briefly recall the definition and main properties of generalized mixture operators. They extend the classical weighted averaging operators (which remain a special case) in the sense that the classical constant weights become weighting functions defined on the aggregation domain. In other words, the weights are no longer constant but depend on the values of the aggregation variables. The construction is analogous to the one in [1] [2], where mixture operators and generalized mixture operators have been originally introduced.

Consider n variables in the unit interval $x_i \in [0,1]$, with $i = 1, \dots, n$, and an equal number of weights $w_i \geq 0$ satisfying the normalization condition $\sum_{j=1}^n w_j = 1$. The vectors containing the n variables and the n weights are denoted $\mathbf{x} = [x_i]$ and $\mathbf{w} = [w_i]$, respectively.

As mentioned before, the classical weighted averaging operator is defined as $WA(\mathbf{x}) = \sum_{i=1}^n w_i x_i$, where $\mathbf{x} = [x_i]$ encodes the various attribute satisfaction values of an alternative.

The weighted averaging operator $WA(\mathbf{x})$ is linear, differentiable, monotonic, and compensative. For what concerns monotonicity and compensation, it means that $\partial_i WA(\mathbf{x}) \geq 0$ for $i = 1, \dots, n$ and $\min(\mathbf{x}) \leq WA(\mathbf{x}) \leq \max(\mathbf{x})$, respectively. Moreover, if all the weights are positive $w_i > 0$, the WA operator is strictly monotonic, i.e. $\partial_i WA(\mathbf{x}) > 0$ for $i = 1, \dots, n$, and strictly compensative, i.e. $\min(\mathbf{x}) < WA(\mathbf{x}) < \max(\mathbf{x})$, except when all the variables x_i coincide, in which case $\min(\mathbf{x}) = WA(\mathbf{x}) = \max(\mathbf{x})$.

Consider now n functions $f_i(x)$, with $i = 1, \dots, n$, defined in the unit interval, for $x \in [0,1]$. We assume that these functions are of class \mathcal{C}^1 , i.e. each function $f_i(x)$ is continuous and has a continuous derivative $f_i'(x)$ in the unit interval, for $i = 1, \dots, n$.

Furthermore, we assume that the functions $f_i(x)$ are positive in the unit interval,

Condition (I) : $f_i(x) > 0$ for $i = 1, \dots, n$ and $x \in [0,1]$,

and also that their derivatives $f_i'(x)$ are non-negative in the unit interval,

Condition (II) : $f_i'(x) \geq 0$ for $i = 1, \dots, n$ and $x \in [0,1]$.

The generalized mixture operator $W(\mathbf{x})$ generated by the n functions $f_i(x)$ is defined as

$$W(\mathbf{x}) = \frac{\sum_{i=1}^n f_i(x_i)x_i}{\sum_{j=1}^n f_j(x_j)}.$$

Mixture operators are non linear, differentiable, and compensative operators, but they are not necessarily monotonic. Nevertheless, interesting sufficient conditions for monotonicity can be derived, as indicated below.

The generalized mixture operator $W(\mathbf{x})$ can also be written as a form of weighted average, in which weighting functions have replaced the classical constant weights,

$$W(\mathbf{x}) = \sum_{i=1}^n w_i(\mathbf{x})x_i \quad \text{with weighting functions } w_i(\mathbf{x}) = \frac{f_i(x_i)}{\sum_{j=1}^n f_j(x_j)}.$$

The positive weighting functions $w_i(\mathbf{x}) > 0$ satisfy the normalization condition $\sum_{j=1}^n w_j(\mathbf{x}) = 1$.

In [1] [2], the authors have derived a general sufficient condition for the strict monotonicity of the mixture operator $W(\mathbf{x})$, which means $\partial_i W(\mathbf{x}) > 0$ for $i = 1, \dots, n$. In the unit interval, and for positive generating functions with non-negative derivatives, the sufficient condition takes the following simple form,

Condition (III) : $f_i'(x) \leq f_i(x)$, for $i = 1, \dots, n$ and $x \in [0,1]$.

In the paper we call this the *monotonicity condition*.

Notice that any of the three conditions (I-III) holds independently for each $i = 1, \dots, n$. We shall make use of this fact in what follows by applying the three conditions (I-III) to a *general* linear or quadratic weight generating function $f(x)$, with only non-classical parameters β and γ regarding the weight dependence on the attribute satisfaction values. After that we consider the associated effective weight generating function $F(x) = \alpha f(x) / f(1)$, whose value for unit attribute satisfaction $F(1) = \alpha$ is given by the classical parameter α , representing the a priori importance of the attribute.

The following sections of the paper describe two different ways in which generalized mixture operators can model weighted aggregation in multiple attribute decision-making. In section 3 we use *linear* weight generating functions and in section 4 we use *quadratic* weight generating functions. Naturally, the former weight generating functions are a special case of the latter.

4 Linear weight generating functions

In this section we consider weight generating functions of the linear type and derive the implications of conditions (I), (II), (III) in the linear case.

Consider the linear weight generating function $l(x) = 1 + \beta x$, whose derivative is $l'(x) = \beta$. The classical weighted averaging operator corresponds to the parametric choice $\beta = 0$.

Theorem 1: In the linear case, with the weight generating function $l(x)$ defined above, conditions (I), (II), (III) hold *if and only if* $0 \leq \beta \leq 1$.

Proof : The parametric condition $\beta \geq 0$ is sufficient for $l(x) > 0$ for $x \in [0,1]$ and is necessary and sufficient for $l'(x) \geq 0$ for $x \in [0,1]$.

On the other hand, the monotonicity condition $l'(x) \leq l(x)$ for $x \in [0,1]$ can be written in the equivalent form $\beta(1-x) \leq 1$ for $x \in [0,1]$.

This condition can be decomposed in the border conditions $x = 0 : \beta \leq 1$ and $x = 1 : 0 \leq 1$ (the latter is an identity), plus the condition $\beta \leq 1/(1-x)$ in the open unit interval $x \in (0,1)$.

The condition in the open unit interval is weaker than the first border condition, and can thus be ignored. We conclude that the monotonicity condition $l'(x) \leq l(x)$ for $x \in [0,1]$ is equivalent to the parametric condition $\beta \leq 1$. ♦

The effective weight generating function $L(x)$ associated with $l(x)$ is given by

$$L(x) = \alpha \frac{l(x)}{l(1)} = \alpha \frac{1 + \beta x}{1 + \beta}$$

with parameters $0 < \alpha \leq 1$ and $0 \leq \beta \leq 1$.

The extreme values $0 < L(0) \leq L(1) \leq 1$ of the effective weight generating function $L(x)$ are given by $L(0) = \alpha/(1 + \beta)$ and $L(1) = \alpha$. The parametric choice $0 < \alpha \leq 1$ controls the value $L(1) = \alpha$ of the effective weight generating function, when the attribute satisfaction value is one.

On the other hand, the parametric choice $0 \leq \beta \leq 1$ controls the ratio $L(1)/L(0) = 1 + \beta$ between the largest and smallest values of the effective weight generating function, when the attribute satisfaction values are zero and one. In this respect, the parametric condition $0 \leq \beta \leq 1$ can be written as $1 \leq L(1)/L(0) \leq 2$, which means that the ratio $L(1)/L(0)$ is at most 2 in the linear case.

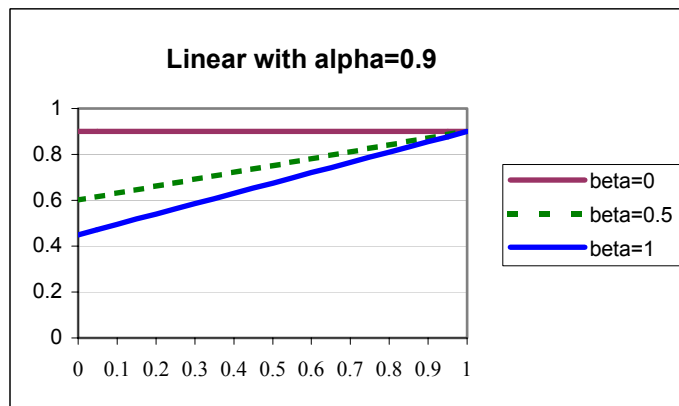


Figure 1. Effective weight generating functions: linear case

Figure 1 above shows the plots of three linear effective weight generating functions, associated with $\alpha = 0.9$ and three distinct values of parameter β . Observe that parameter α controls the value of the effective weight generating function when the degree of attribute satisfaction is 1, while parameter β controls the lower limit of the effective weight generating function.

5 Quadratic weight generating functions

In this section we consider weight generating functions of the quadratic type and derive the implications of conditions (I), (II), (III) in the quadratic case. Clearly, the linear case can be obtained as a special case of the more general quadratic case.

Consider the quadratic weight generating function $q(x) = 1 + (\beta - \gamma)x + \gamma x^2$, whose derivative is given by $q'(x) = \beta - \gamma(1 - 2x)$. The linear case discussed in the previous section corresponds to the parametric choice $\gamma = 0$.

In this case we assume the general parametric condition $\gamma \geq 0$ so as to consider only convex quadratic weight generating functions. The reason is that convex functions are more natural than concave functions in representing conditions (I), (II), (III). In fact, the two functions $q(x) > 0$ and $q'(x) \geq 0$ are comonotonically non-decreasing in the convex case, and this is compatible with the form of monotonicity condition $q'(x) \leq q(x)$. In the concave case, instead, $q(x) > 0$ would still be non-decreasing but $q'(x) \geq 0$ would now be non-increasing.

Theorem 2: In the quadratic case, with the weight generating function $q(x)$ defined above, plus the convexity assumption $\gamma \geq 0$, conditions (I), (II), (III) hold *if and only if* $0 \leq \gamma \leq 1$ and $\gamma \leq \beta \leq \beta_c(\gamma)$, where the critical beta function $\beta_c(\gamma)$ is defined as

$$\beta_c(\gamma) = \gamma + 1 \quad \text{for } 0 \leq \gamma \leq 0.5 \quad \text{and} \quad \beta_c(\gamma) = \gamma + 2\sqrt{\gamma(1-\gamma)} \quad \text{for } 0.5 \leq \gamma \leq 1.$$

The graphical plot of the critical beta function $\beta_c(\gamma)$ is shown in figure 2.

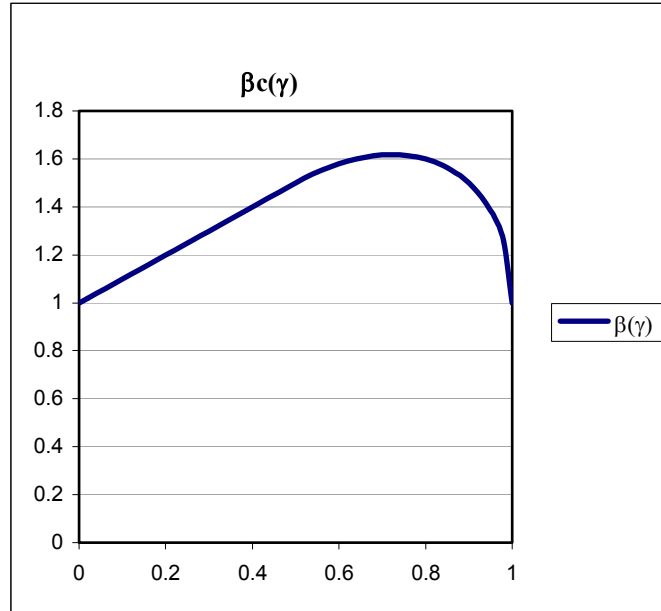


Figure 2. Critical beta function

Proof : Assuming the general parametric condition $\gamma \geq 0$, the parametric condition $\beta \geq \gamma$ is sufficient for $q(x) > 0$ for $x \in [0,1]$ and is necessary and sufficient for $q'(x) \geq 0$ for $x \in [0,1]$.

The monotonicity condition $q'(x) \leq q(x)$ for $x \in [0,1]$ can be written as

$$\beta(1-x) \leq \gamma(1-x)(2-x) + 1 - \gamma \text{ for } x \in [0,1].$$

This condition can be decomposed in the border conditions $x = 0 : \beta \leq \gamma + 1$ and $x = 1 : \gamma \leq 1$, plus the condition $\beta \leq \gamma(2-x) + (1-\gamma)/(1-x)$ in the open interval $x \in (0,1)$.

Up to now, the explicit parametric conditions are $0 \leq \gamma \leq 1$ and $\gamma \leq \beta \leq \gamma + 1$. The condition in the open unit interval $x \in (0,1)$ can be studied as follows.

Define the function $\Gamma(x) = \gamma(2-x) + (1-\gamma)/(1-x)$ on the open unit interval $x \in (0,1)$. Its first and second derivatives are given by $\Gamma'(x) = -\gamma + (1-\gamma)/(1-x)^2$ and $\Gamma''(x) = 2(1-\gamma)/(1-x)^3$. The function $\Gamma(x)$ is strictly convex for $0 \leq \gamma < 1$ and linear for $\gamma = 1$.

Case $0 \leq \gamma \leq 0.5$: in this case $\Gamma'(x) > 0$, and therefore the condition on the open unit interval is weaker than the first border condition. The resulting condition is $\beta \leq \gamma + 1$.

Case $0.5 < \gamma < 1$: in this case the function $\Gamma(x)$ has a minimum value corresponding to $x_0 = 1 - \sqrt{(1-\gamma)/\gamma}$, with $x_0 \in (0,1)$. Substituting back in the function, we obtain the minimum value $\Gamma(x_0) = \gamma + 2\sqrt{\gamma(1-\gamma)}$. The resulting condition is $\beta \leq \gamma + 2\sqrt{\gamma(1-\gamma)}$.

Case $\gamma = 1$: in this case one must study directly the monotonicity condition in the open unit interval, in the form $\beta \leq 2 - x$ for $x \in (0,1)$. The resulting (limiting) condition is $\beta \leq 1$.

We conclude that the monotonicity condition $l'(x) \leq l(x)$ for $x \in [0,1]$ is equivalent to the parametric condition $\beta \leq \beta_c(\gamma)$ where

$$\beta_c = \gamma + 1 \text{ for } 0 \leq \gamma \leq 0.5 \quad \text{and} \quad \beta_c = \gamma + 2\sqrt{\gamma(1-\gamma)} \text{ for } 0.5 \leq \gamma \leq 1.$$

Notice that the monotonicity condition in the second half of the unit interval is stronger than the one in the first half, that is $\gamma + 2\sqrt{\gamma(1-\gamma)} \leq \gamma + 1$ for $0 \leq \gamma \leq 1$, and the equality holds only for $\gamma = 0.5$.

Summarizing, the set of three conditions (I-III) is equivalent to the parametric conditions $0 \leq \gamma \leq 1$ and $\gamma \leq \beta \leq \beta_c(\gamma)$. ♦

The effective weight generating function $Q(x)$ associated with $q(x)$ is given by

$$Q(x) = \alpha \frac{q(x)}{q(1)} = \alpha \frac{1 + (\beta - \gamma)x + \gamma x^2}{1 + \beta}$$

with parameters $0 \leq \gamma \leq 1$, $\gamma \leq \beta \leq \beta_c(\gamma)$, and $0 < \alpha \leq 1$.

The extreme values $0 < Q(0) \leq Q(1) \leq 1$ of the effective weight generating function $Q(x)$ are $Q(0) = \alpha/(1 + \beta)$ and $Q(1) = \alpha$. Once more, the parametric choice $0 < \alpha \leq 1$ controls the value $Q(1) = \alpha$ of the effective weight generating function, when the attribute satisfaction degree is one.

On the other hand, the parametric condition $0 \leq \gamma \leq 1$ controls the amount of non linearity in the effective weight generating function, and the crucial parametric condition $\gamma \leq \beta \leq \beta_c(\gamma)$ controls the ratio $Q(1)/Q(0) = 1 + \beta$ between the largest and smallest values of the effective weight generating function, when the attribute satisfaction degrees are zero and one. In this respect, the parametric condition $\gamma \leq \beta \leq \beta_c(\gamma)$ can be written as $1 + \gamma \leq Q(1)/Q(0) \leq 1 + \beta_c(\gamma)$.

Now, the maximum value of $\beta_c(\gamma)$ corresponds to $\gamma_0 = (1 + \sqrt{5}) / 2\sqrt{5} = 0.72367 \dots$ and is given by $\beta_c(\gamma_0) = (1 + \sqrt{5}) / 2 = 1.61803 \dots$ as illustrated by the plot in figure 2. This means that in the quadratic case the ratio $Q(1)/Q(0)$ (at most 2.618...) can be significantly larger than the corresponding ratio $L(1)/L(0)$ in the linear case (at most 2).

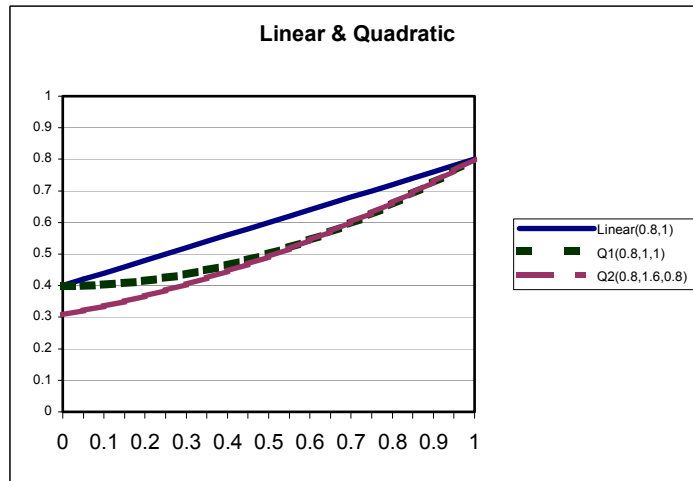


Figure 3. Effective weight generating functions: linear and quadratic cases

Figure 3 depicts the plots of a linear and two quadratic weighting generating functions, with a common value of $\alpha = 0.8$. As mentioned before, parameter α controls the largest value of the weight generating functions, $L(1) = \alpha$ and $Q(1) = \alpha$. Parameter β , the crucial one, controls the ratio $L(1)/L(0) = 1 + \beta$ and $Q(1)/Q(0) = 1 + \beta$ between the largest and the smallest values of the effective weight generating functions. It takes the values 1 (linear and quadratic Q1) and 1.6 (quadratic Q2). It is therefore responsible for most of the non-linearity present in the weighting functions and generalized mixture operators.

Finally, parameter γ controls the curvature of the quadratic effective weight generating functions and takes the values 1 (Q1) and 0.8 (Q2). As can be seen in the figure, both quadratic cases assign less weight to intermediate attribute satisfaction values than the linear case. But most of all, the quadratic Q2 case *improves significantly* the ratio between the upper and lower values of the effective weight generating function. In other words, parameter γ serves to *potentiate* the effect of parameter β in the quadratic case, which is the main reason why the more general quadratic case is in fact *more interesting* theoretically and practically than the linear case.

Another way of illustrating the continuous non-linear effect present in the preference relations obtained using the generalized mixture operators considered in this paper is to draw the corresponding level curves, as below. Figure 4 refers to the linear and quadratic weight generating functions, whereas figure 5 refers to the constant weight generating function, i.e. the classical WA operator. In all three cases, the a priori importances of the two attributes have been considered equal (parameter $\alpha = 1$) so that the differences in the preference relations are produced exclusively by the parameters β and γ (parameters $\beta = \gamma = 1$).

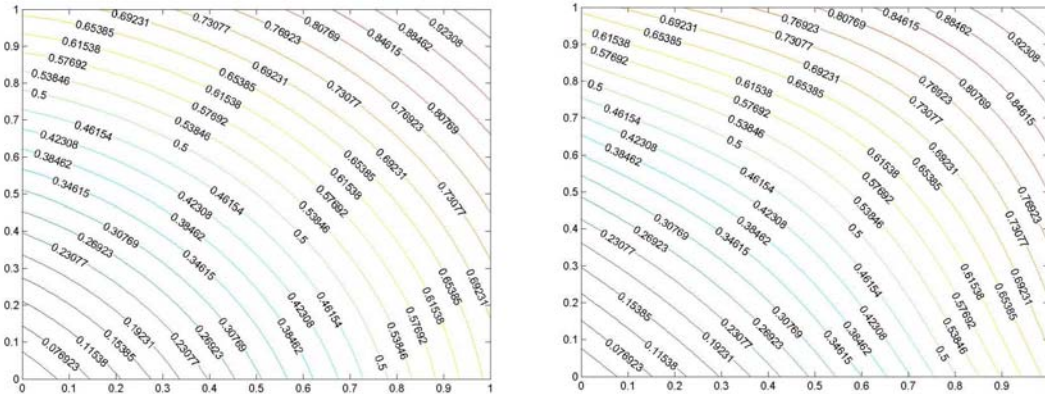


Figure 4 Level curves: linear and quadratic cases

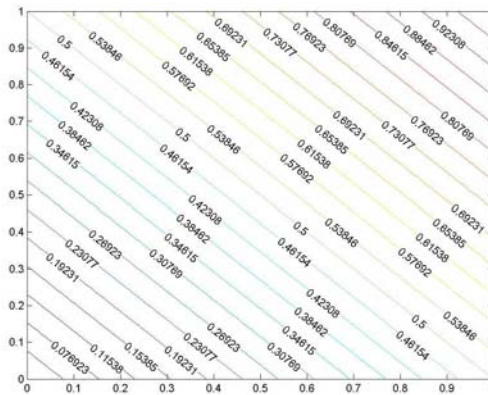


Figure 5 Level curves: classical WA case

6 An illustrative example

In this example we shall consider two alternatives and two attributes (criteria) with the following starting attribute satisfaction values (table 2):

Attributes	C1	C2
Alternative 1	0.2	0.4
Alternative 2	0.3	0.2

Table 2

In order to solve the above problem we use linear and quadratic weight generating functions plus the weighted average (WA). Both linear and quadratic weight generating functions were studied in detail in sections 4 and 5. Hence, here we only need to define the decisor preferences in terms of the a priori importance of each attribute (parameter α) and a set of values for parameters β and γ . The values used for the latter two parameters are based on their critical values as discussed in the previous sections. In summary, the set of parameters and preferences used are depicted in table 3.

Parameters	α	β	γ
Linear	0.8	1	
Quadratic (Q1)	0.8	1	1
Quadratic (Q2)	0.8	1.6	0.8
Weighted average	0.8		

Table 3

As can be observed in table 3, we considered equally a priori importances (parameter α) for each attribute, with a value of “Important” which corresponds to a level of 0.8. This simplification was considered to facilitate the discussion of the results obtained.

Now, we will show the details of the calculations to obtain a classification for both alternatives, using the above problem (table 1). For reasons of space we only show the calculations for two weight generating functions: a linear function and a quadratic function (Q2, table2), which use the respective parameters depicted in table 3. The calculations can be divided in three different steps:

1. Calculation of the pre-weights based on the weight generating functions proposed.

$$\text{Linear}(\text{alt1}, C1) = (0.8 \times (1 \times 0.2 + 1)) / (1 + 1) = 0.48$$

$$\text{Linear}(\text{alt1}, C2) = (0.8 \times (1 \times 0.4 + 1)) / (1 + 1) = 0.56$$

$$\text{Linear}(\text{alt2}, C1) = (0.8 \times (1 \times 0.3 + 1)) / (1 + 1) = 0.52$$

$$\text{Linear}(\text{alt2}, C2) = (0.8 \times (1 \times 0.2 + 1)) / (1 + 1) = 0.48$$

$$Q2(alt1, C1) = ((0.8 \times (0.8 \times 0.2^2) + (1.6 - 0.8) \times 0.2) + 1) / (1 + 1.6) = 0.37$$

$$Q2(alt1, C2) = ((0.8 \times (0.8 \times 0.4^2) + (1.6 - 0.8) \times 0.4) + 1) / (1 + 1.6) = 0.45$$

$$Q2(alt2, C1) = ((0.8 \times (0.8 \times 0.3^2) + (1.6 - 0.8) \times 0.3) + 1) / (1 + 1.6) = 0.4$$

$$Q2(alt2, C2) = ((0.8 \times (0.8 \times 0.2^2) + (1.6 - 0.8) \times 0.2) + 1) / (1 + 1.6) = 0.37$$

2. Normalization of pre-weights to obtain normalized weights.

$$Linear(alt1, C1) = 0.48 / (0.48 + 0.56) = 0.46$$

$$Linear(alt1, C2) = 0.56 / (0.48 + 0.56) = 0.54$$

$$Linear(alt2, C1) = 0.52 / (0.52 + 0.48) = 0.52$$

$$Linear(alt2, C2) = 0.48 / (0.52 + 0.48) = 0.48$$

$$Q2(alt1, C1) = 0.37 / (0.37 + 0.45) = 0.45$$

$$Q2(alt1, C2) = 0.45 / (0.37 + 0.45) = 0.55$$

$$Q2(alt2, C1) = 0.4 / (0.4 + 0.37) = 0.52$$

$$Q2(alt2, C2) = 0.37 / (0.4 + 0.37) = 0.48$$

3. Final results for each alternative.

$$Linear(alt1) = 0.46 \times 0.2 + 0.54 \times 0.4 = 0.308$$

$$Linear(alt2) = 0.52 \times 0.3 + 0.48 \times 0.2 = 0.252$$

$$Q2(alt1) = 0.45 \times 0.2 + 0.55 \times 0.4 = 0.310$$

$$Q2(alt2) = 0.52 \times 0.3 + 0.48 \times 0.2 = 0.252$$

In order to make a comparison between different weighting schemes we solved the initial problem (table 2) considering increases of 0.1 for the first attribute until it reaches 1, and the second attribute is not changed throughout the various cases. We start with Case 1, depicted in table 1, and we go on increasing attribute 1 (C1) by 0.1 until Case 8 (table 4) has the values 0.9 and 1 for alternatives 1 and 2, respectively. Table 4 shows the final results obtained, for the linear weight generating function (denoted Linear), plus for both quadratic functions, Q1 and Q2 (parameters for the Linear and Quadratic are defined in table 3), and also the classical weighted average (WA). It should be stressed that we use the name Linear because the weight generating function is linear, not because the weighting function is linear (see details in section 4). In addition, to improve readability, the best results for each case are highlighted in table 4.

FINAL RESULTS

		C1	C2	Linear	Q1	Q2	WA
Case 1	Alt1	0.2	0.4	0.308	0.305	0.310	0.300
	Alt2	0.3	0.2	0.252	0.251	0.252	0.250
Case 2	Alt1	0.3	0.4	0.352	0.352	0.352	0.350
	Alt2	0.4	0.2	0.308	0.305	0.310	0.300
Case 3	Alt1	0.4	0.4	0.400	0.400	0.400	0.400
	Alt2	0.5	0.2	0.367	0.364	0.372	0.350
Case 4	Alt1	0.5	0.4	0.452	0.452	0.452	0.450
	Alt2	0.6	0.2	0.429	0.427	0.439	0.400
Case 5	Alt1	0.6	0.4	0.507	0.508	0.510	0.500
	Alt2	0.7	0.2	0.493	0.494	0.510	0.450
Case 6	Alt1	0.7	0.4	0.565	0.569	0.572	0.550
	Alt2	0.8	0.2	0.560	0.567	0.586	0.500
Case 7	Alt1	0.8	0.4	0.625	0.634	0.639	0.600
	Alt2	0.9	0.2	0.629	0.645	0.666	0.550
Case 8	Alt1	0.9	0.4	0.688	0.705	0.710	0.650
	Alt2	1	0.2	0.700	0.726	0.749	0.600

Table 4

Observing table 4 we see that, for the first 4 cases, all approaches consider as best result alternative 1 (Alt1). The reason for this is that the dominance effect of attribute C2 is bigger than the dominance effect of attribute C1, when considering alternative 1 versus alternative 2. Now, observing case 5, since the satisfaction values of attribute C1 reached 0.6 and 0.7 we see that the quadratic function Q2 gives the same result for both alternatives, i.e. the increase in attribute C1 for alternative 2 is sufficient to overcome the lower satisfaction value for C2, i.e. the dominance effect is changing. For case 6, Q2 already considers that alternative 2 is better than alternative 1, i.e. the dominance effect changed, hence making it the most sensitive weight generating function towards attribute satisfaction levels. In cases 7 and 8 all proposed weighting functions consider that alternative 2 is the best, i.e. the value of attribute C1 in alternative 2 is sufficiently big to change the dominance effect of C2 over C1 in all proposed functions. Moreover, it is quite visible in case 7 that the quadratic weight generating functions are more sensitive than the linear one when there are increases in the satisfaction values of the attributes, because the difference between results for alternative 1 and 2 is bigger in both the quadratic functions.

In addition, note that in the classical weighted average (WA) alternative 1 is always selected because, when the weights are identical, the distinguishable feature is just the total aggregation value, i.e. the

weighted average fails to distinguish between high satisfaction values and lower ones – clearly a limitation. Furthermore, the WA does not take into account the dominance effect of considering that higher satisfaction values of the criteria should influence the final decision function result.

To conclude this illustrative example we will discuss the normalized weights that correspond to the results presented above (table 4). For reasons of space, table 5 only depicts the normalized weights (calculated as described above) for cases 3, 4, 5, 6, 7, 8 because they are the most relevant.

		Case 3		Case 4		Case 5		Case 6		Case 7		Case 8	
		C1	C2	C1	C2	C1	C2	C1	C2	C1	C2	C1	C2
L	alt1	0.50	0.50	0.52	0.48	0.53	0.47	0.55	0.45	0.56	0.44	0.58	0.42
	alt2	0.56	0.44	0.57	0.43	0.59	0.41	0.60	0.40	0.61	0.39	0.63	0.38
Q1	alt1	0.50	0.50	0.52	0.48	0.54	0.46	0.56	0.44	0.59	0.41	0.61	0.39
	alt2	0.55	0.45	0.57	0.43	0.59	0.41	0.61	0.39	0.64	0.36	0.66	0.34
Q2	alt1	0.50	0.50	0.52	0.48	0.55	0.45	0.57	0.43	0.60	0.40	0.62	0.38
	alt2	0.57	0.43	0.60	0.40	0.62	0.38	0.64	0.36	0.67	0.33	0.69	0.31
WA	alt1	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50
	alt2	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50

Table 5

Observing table 5, it is clear that while normalized weights are constant in the classical weighted average, in the weighting functions proposed in this paper the normalized weights have a highly non linear behaviour, which goes beyond the simple (linear or quadratic) functional form of the generating functions. This is due to the compensatory nature of the weight normalization scheme and reflects the difficulties related with monotonicity discussed in the previous sections.

7 Conclusions

In this paper we investigate two different ways in which the weights can depend on the satisfaction degrees of the various attributes (criteria). We have proposed and discussed linear and quadratic weight generating functions, that penalize poorly satisfied attributes and reward well satisfied attributes. We have performed an in depth analysis of both functions, particularly the parametric conditions required by the monotonicity properties.

We have illustrated the weighting functions behaviour with an example and compared the results with the classical weighted average scheme. The approach proposed in this paper shows clearly a richer range

of possible aggregation schemes in which the weights take into account the attribute satisfaction values of the various attributes.

The generalized mixture operators considered in this paper are generated by increasing weight generating functions. For this reason the resulting aggregation is of the OR-type, i.e. it is characterized by orness values in the interval $[0.5,1]$. It is our plan for future work to investigate the opposite case, with decreasing weight generating functions.

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